BMB/Bi/Ch 173 – Winter 2017

Homework Set 1.1 – Assigned 1-10-17, Due 1-17-17 by 10:30am

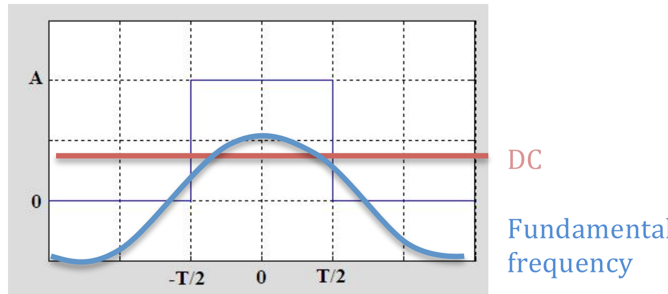
TA: Wen Zhou (201 Kerckoff, office hour: Fri 1/13 5-6pm, Mon 1/16 by appointment)

80 points total

1. (30 points) **Fourier Transforms and convolution theorem**

I. (8 points) Sketch the DC component and fundamental frequency of the Fourier Transform for the rectangular wave below. Write the mathematical description of the wave.

$$\Pi(t) = \begin{cases} A, & |t| \le T/2 \\ 0, & otherwise \end{cases}$$



II. (12 points) The Fourier Transform of the rectangular wave is a sinc function. The derivation is shown below:

$$\mathcal{F}\{g(t)\} = G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi i f t}\,dt$$

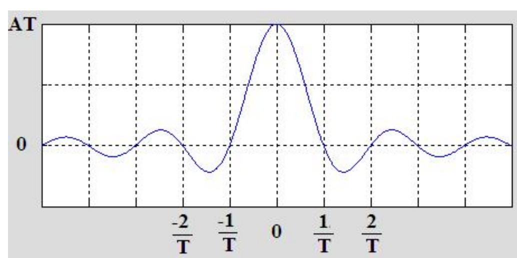
$$= \int_{-T/2}^{T/2} Ae^{-2\pi i f t}\,dt = \frac{A}{-2\pi i f}\left[e^{-2\pi i f t}\right]_{-T/2}^{T/2}$$

$$= \frac{A}{-2\pi i f}\left[e^{-\pi i f T} - e^{\pi i f T}\right] = \frac{AT}{\pi f T}\left[\frac{e^{\pi i f T} - e^{-\pi i f T}}{2i}\right]$$

$$= \frac{AT}{\pi f T}\sin(\pi f T) = AT\left[\mathrm{sinc}(fT)\right]$$

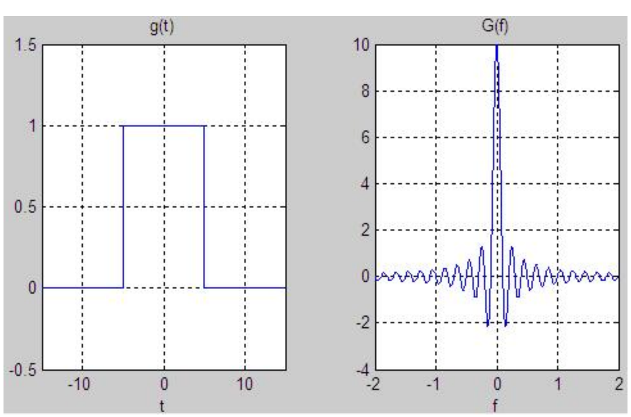
The sinc function is defined by $\mathrm{sinc}(t) = \dfrac{\sin(\pi t)}{\pi t}$

And the Fourier Transform of the wave can be plot as

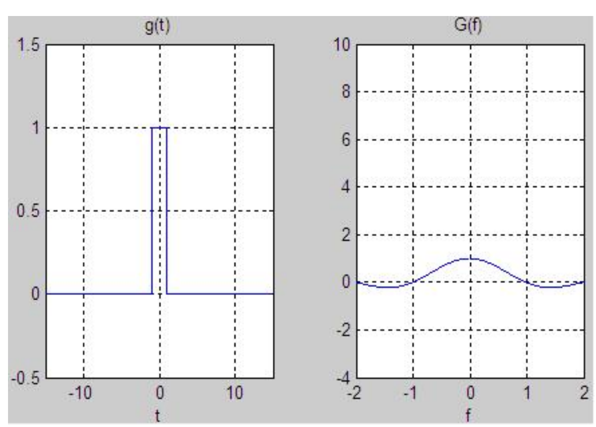


To understand the waves better, sketch and compare the rectangular pulses defined for T=10, and T=1, and their corresponding Fourier Transformations, with amplitude A=1. (To help compare, keep the Y axis scale the same for the two T values) What do you learn from this comparison?

T=10

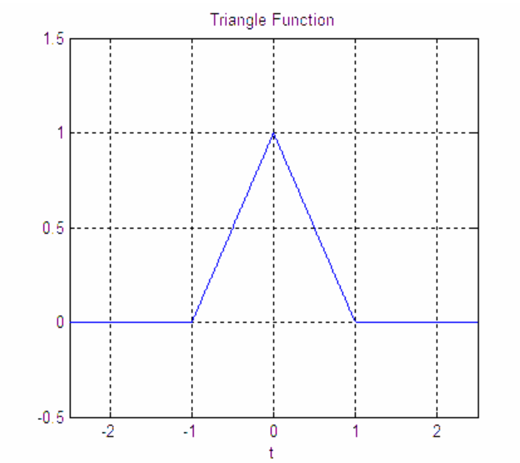


T=1



The wider square pulse requires more high frequency components and a much higher amplitude in the Fourier transform than the narrower square wave. (Other reasonable answers also accepted as long as you compared the amplitude and high frequency components of the two FTs)

III) (10 points) The convolution of two of the rectangular pulses is a triangle pulse. For the triangle wave that is given below, what is its Fourier Transform? (Hint: apply the convolution theorem)



Convolution theorem solution:

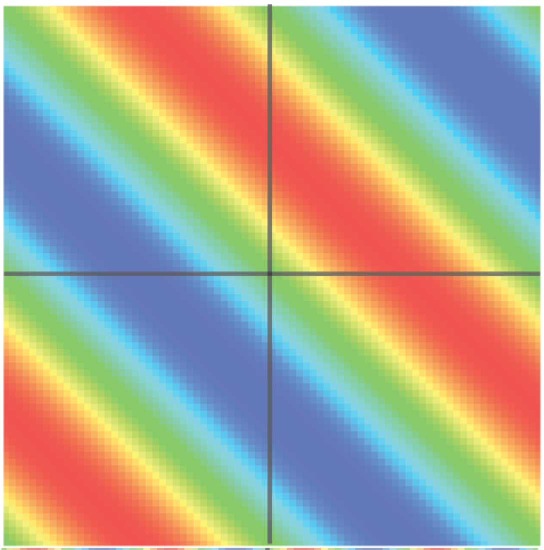
$$\Lambda(t) = \Pi(t) * \Pi(t)$$

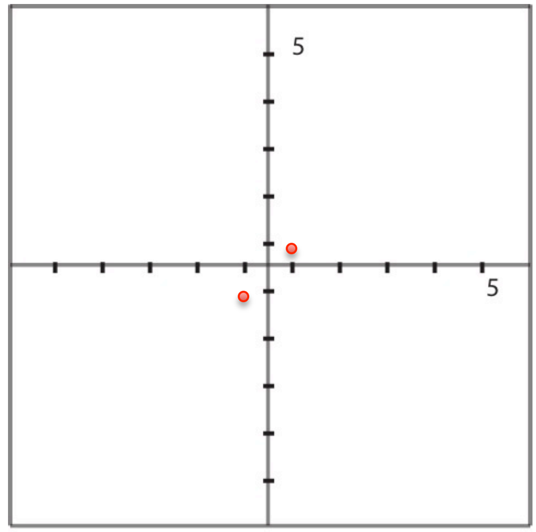
$$\begin{aligned}\Delta(f) &= \Im\{\Lambda(t)\} = \Im\{\Pi(t) * \Pi(t)\} \\ &= \Im\{\Pi(t)\}\Im\{\Pi(t)\} \\ &= \mathrm{sin}c(f)\cdot \mathrm{sin}c(f) \\ &= \mathrm{sin}c^2(f)\end{aligned}$$

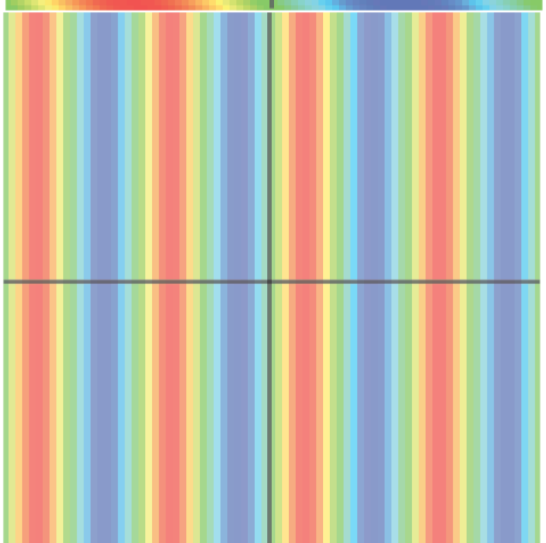
Brute-force solution: (not encouraged but accepted)

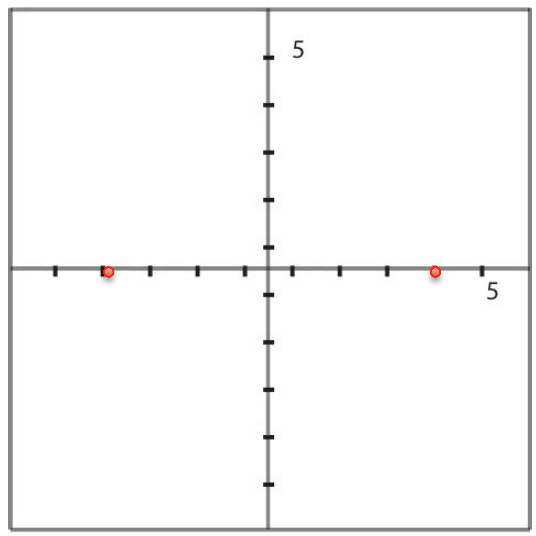
$$\begin{aligned}\Delta(f) = \Im\{\Lambda(t)\} &= \int_{-\infty}^{\infty} \Lambda(t) e^{-2\pi i f t}\, dt \\ &= \int_{-1}^{0} (1+t) e^{-2\pi i f t}\, dt + \int_{0}^{1} (1-t) e^{-2\pi i f t}\, dt \\ &= \left[\frac{1+2\pi i f}{4\pi^2 f^2} - \frac{e^{2\pi i f}}{4\pi^2 f^2}\right] - \left[\frac{2\pi i f - 1}{4\pi^2 f^2} + \frac{e^{-2\pi i f}}{4\pi^2 f^2}\right] \\ &= -\frac{e^{-2\pi i f}\left(e^{2\pi i f} - 1\right)^2}{4\pi^2 f^2} \\ &= -\frac{e^{-2\pi i f}\left(e^{\pi i f}\left[e^{\pi i f} - e^{-\pi i f}\right]\right)^2}{4\pi^2 f^2} \\ &= -\frac{e^{-2\pi i f} e^{2\pi i f} (2i)^2 \sin^2(\pi f)}{4\pi^2 f^2} \\ &= \left(\frac{\sin(\pi f)}{\pi f}\right)^2 = \mathrm{sin}c^2 f\end{aligned}$$

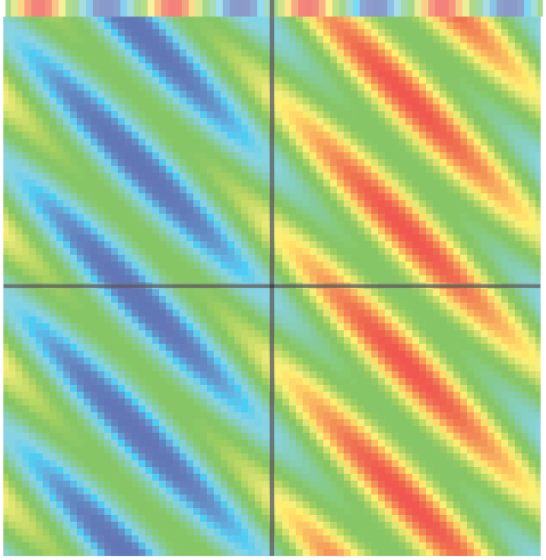
2. (18 Points) **2D Sine Waves and Reciprocal Space**
For each 2D sine wave or sum of 2D sine waves on the left, draw the fourier transform on the h,k axes on the right. Don't worry about phase.

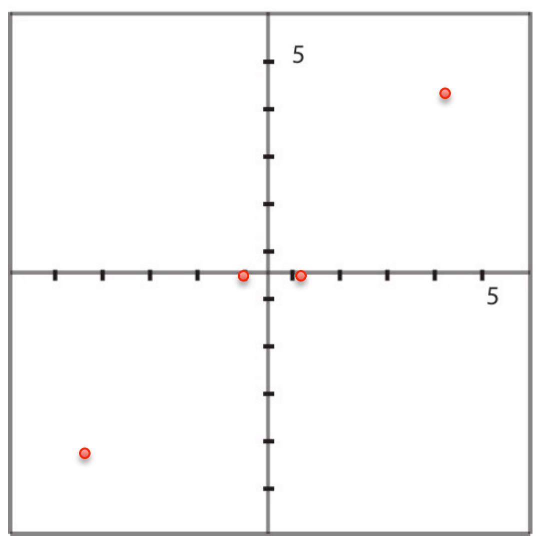






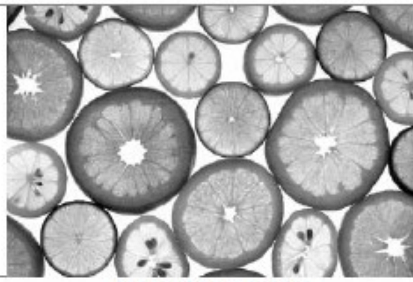




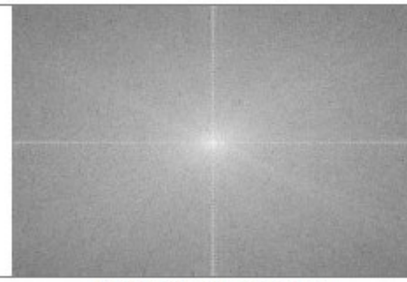


3. (12 Points) **Image Filtering**

Below are an image and its Fourier transform:

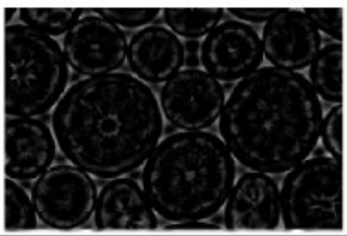


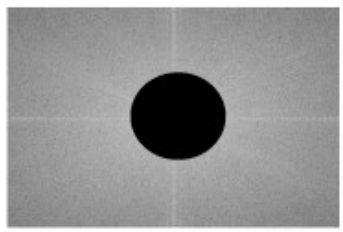
Original Image



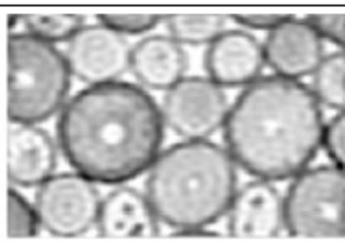
Fourier Transform

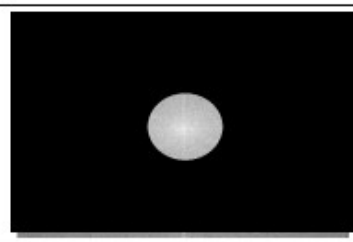
For each of the images below, qualitatively draw where the filter was in the Fourier space by shading in the frequencies removed by the filter. Explain your reasoning.



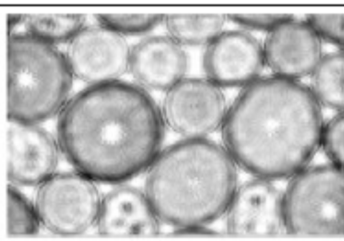


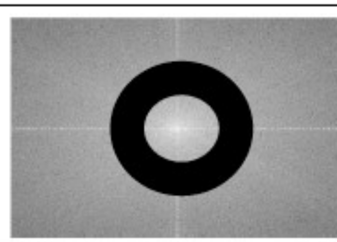
High pass





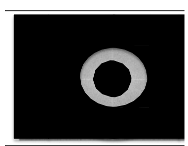
Low pass





Band pass

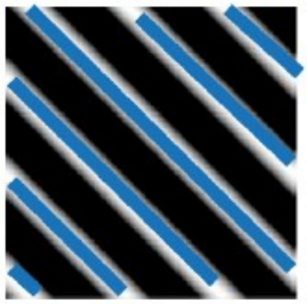
Or



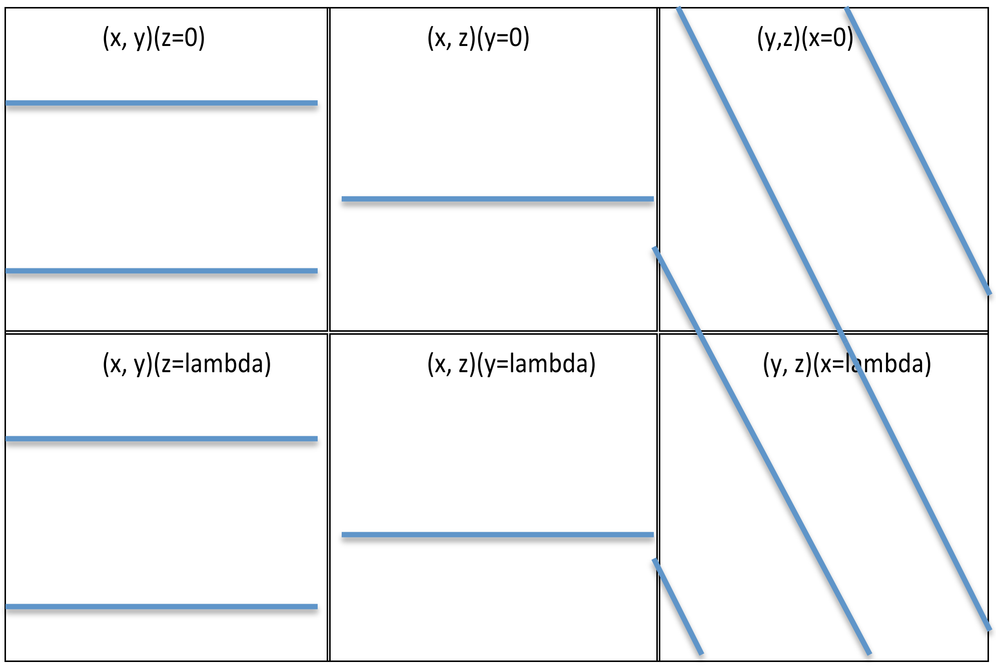
The 1st image results from high-pass filter. The filter preserves the sharp crisp edges from the original but loses the larger regions of dark and bright.

4. (20 Points) **3D waves**

You can draw a line over the maxima of a 2D sine wave this way



a



b) Now copy the lines you drew to the faces of the cube below. Describe qualitatively the shape and form of the maxima of the 3D sine wave in the cube.